

The Lee Fields Medal VIII

Time Allowed: Up to Two Hours and 15 Minutes

Tables and calculators may be used. Each question is worth 10 Marks.

1. In 1837, Dirichlet proved the following famous theorem about prime numbers:

   Dirichlet's Theorem. *Let a and d be positive whole numbers with no common factor greater than* 1. *Then there are infinitely many prime numbers of the form*

   $$a + nd \quad ; \qquad (\text{where } n \text{ is a natural number}) .$$

   Use Dirichlet's Theorem to prove that there are infinitely many primes p such that both $p - 2$ and $p + 2$ are composite.

2. Three non-zero complex numbers z_1, z_2, z_3 are in geometric progression and satisfy

   $$z_1 + z_2 + z_3 = 0.$$

   By finding the possible value of the common ratio r, or otherwise, show that r has non-zero imaginary part.

3. Given that $0 < b < a$ and $a^2 + b^2 = 6ab$, what is the value of

   $$\frac{a+b}{a-b}?$$

4. Draw the graph of each of the following equations:

   - $xy = 0$,
   - $x^2 + y^2 = 0$,
   - $x^2 - y^2 = 0$,
   - $x^3 - y^3 = 0$.

5. How many different triangles with integer side lengths have perimeter 30?

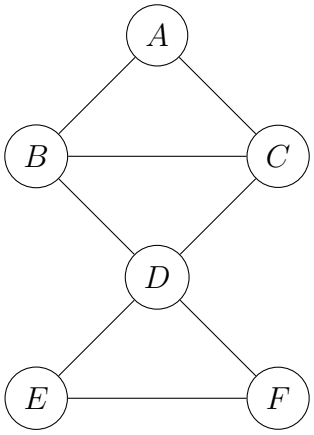
6. Let ABC be a triangle right-angled at C, and let CH be the perpendicular from the hypotenuse to C.

   Find the lengths of CH, AH, and BH in terms of the side lengths a, b, and c only.

7. Consider the following social network consisting of six individuals

$$A, B, C, D, E, F.$$

   An edge between two vertices represents mutual friendship.



   (a) What is the expected value of the number of friends of a randomly selected person?

   (b) A person is selected uniformly at random, and then one of their friends is chosen uniformly at random. Find the expected number of friends of this friend.

   (c) Which expected value is bigger?

8. A rectangle has its lower two corners at $(-x, 0)$ and $(x, 0)$, and its upper two corners lie on the curve

$$y = 12 - x^2,$$

   where $0 \leq x \leq \sqrt{12}$.

   What is the greatest possible area of the rectangle?

9. Alice has eight aunts and uncles, where by an aunt or uncle we mean a sibling of one of Alice's parents. Each of these aunts and uncles has at most three children.

   (i) Suppose exactly k of Alice's aunts and uncles have three children. Show that Alice can have at most

   $$16 + k$$

   first cousins.

   (ii) Suppose Alice has 20 first cousins. Prove that at least four of her eight aunts and uncles must have three children.

10. On a 5×5 chessboard, how many ways are there for the King to move from the lower-left square to the top-right square?

    The only permitted moves are to adjacent squares moving up, right, and diagonally up and to the right.

